International Journal of Theoretical Physics, Vol. 40, No. 1, 2001

Braided Clifford Algebras as Quantum Deformations[1]

Micho Đurđevich[2]

Received April 17, 2000

We present a general algebraic framework for the study of quantum/braided Clifford algebras. We allow that the quadratic form g on the base vector space $\mathbb{V}$ takes values from a noncommutative algebra Σ. Clifford algebra is understood as a Chevalley–Kähler deformation of the braided exterior algebra built from $\mathbb{V}$, Σ, and the initial braid operator $\sigma: \mathbb{V} \otimes \mathbb{V} \to \mathbb{V} \otimes \mathbb{V}$. The new product is canonically associated to g, σ, and Σ, and it is constructed by applying Rota's and Stein Cliffordization.

1. INTRODUCTION

We present a general formalism of braided Clifford algebras, generalizing a formulation proposed in Đurđevich and Oziewicz (1996). The basic philosophy of this paper is the same as that of Đurđevich and Oziewicz (1996) and Oziewicz (1997)—we shall construct braided Clifford algebras as Chevalley–Kähler deformations of braided exterior algebras (Woronowicz, 1989). In contrast to Đurđevich and Oziewicz (1996), however, we shall allow here the situations when the basic quadratic form, defined on the underlying vector space $\mathbb{V}$, takes values from a noncommutative algebra Σ. A simple example of a quantum Clifford algebra with noncommutative metric coefficients can be found in Lawrynowicz *et al.*, (1994).

It is possible to include in the theory various examples of quantum Clifford algebras and spinors naturally appearing in noncommutative geometry. Our formulation naturally fits into a framework (Oziewicz, 1998) of

[1] This work was supported by the United States–Mexico Foundation for Science.
[2] UNAM, Instituto de Matematicas, Area de la Investigacion Cientifica, Circuito Exterior, Ciudad Universitaria, Mexico City, Mexico; e-mail: micho@matem.unam.mx, http://www.matem.unam.mx/~micho

0020-7748/01/0100-0015$19.50/0

diagrammatic manipulations, including a quantum Dirac operator. Furthermore, various braided Clifford algebras are intrinsically associated with compact quantum groups (Woronowicz, 1987) and quantum principal bundles (Đurđevich, 1996; Đurđevich, 1997), where it is possible (Đurđevich, 2000) to develop a general formalism of spinorial frame structures (including the Clifford algebra bundle and Dirac operator) over quantum spaces.

2. ALGEBRAIC SETUP

Consider a complex finite-dimensional vector space $\mathbb{V}$ equipped with a braid operator $\sigma: \mathbb{V} \otimes \mathbb{V} \to \mathbb{V} \otimes \mathbb{V}$ and an antilinear involution $*: \mathbb{V} \to \mathbb{V}$. The map σ corresponds to the classical transposition and $*$ plays the role of the complex conjugation. The map $*$ extends naturally to a $*$-structure on the tensor algebra $\mathbb{V}^{\otimes}$, and we shall denote by the same symbol $*$ a unique antimultiplicative (unital) antilinear extension on the tensor algebra. We shall assume that $*\sigma = \sigma*$.

Let Σ be a $*$-algebra, and let $g: \mathbb{V} \otimes \mathbb{V} \to \Sigma$ be a linear map. We are going to formalize the idea of a *quantum metric*; it will be allowed that metric coefficients do not commute.

(i) *Braided symmetricity of the metric*, $g\sigma = g$. For this to make any sense, it is necessary that 1 belongs to the spectrum of σ. The motivation for this assumption comes from the classical formalism. It is not necessary for our main constructions to work; however, it is a natural condition when we deal with quantum groups that generalize orthogonal transformations—it prevents 'too many' intertwiners from entering the game.

(ii) *Reality property*, $\forall x, y \in \mathbb{V}$, $g(x, y)^* = g(y^*, x^*)$.

(iii) *Funny* σ*-compatibility,*

$$\begin{aligned} g \otimes_\Sigma g &= (g \otimes_\Sigma g)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})\,(\sigma^{-1} \otimes \sigma)(\mathrm{id} \otimes \sigma^{-1} \otimes \mathrm{id}) \\ g \otimes_\Sigma g &= (g \otimes_\Sigma g)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})\,(\sigma \otimes \sigma^{-1})(\mathrm{id} \otimes \sigma^{-1} \otimes \mathrm{id}) \end{aligned} \tag{1}$$

The above two equations are *equivalent* if we assume that the reality condition holds. This property ensures that g is extendible to the level of appropriate Σ-bimodules.

(iv) *Weak positivity.* We have to assume that Σ is realized by operators in the Hilbert space $H = l^2(\mathbb{Z})$. Since in general these operators will be *unbounded*, we have to take care about the domains. We shall assume that there is an everywhere dense linear subspace $H_0 \subseteq H$ which is a common domain for all the operators from Σ. We shall also assume that the *-structure on Σ is represented as taking formal adjoints of linear operators in H_0 and that there exist cyclic and separating vectors $\Omega \in H_0$ for Σ. A possible candidate for positivity would be

$$\forall x \in \mathbb{V}, \quad g(x^*, x) \geq 0, \quad x = 0 \Leftrightarrow g(x^*, x) = 0$$

The reason why we call this condition 'weak' positivity will become clear after we construct a canonical Σ-bimodule structure over $\mathbb{V}$ and introduce a stronger version of positivity.

(v) *Minimality and Invertibility.* The matrix $g_{ij} = g(\theta_i, \theta_j)$ is invertible in Σ, where $\{\theta_1, \ldots, \theta_d\}$ are basis vectors in $\mathbb{V}$, and $d = \dim(\mathbb{V})$. The algebra Σ is generated by the matrix elements of g and g^{-1}.

(vi) Twisting Σ and $\mathbb{V}$. Let ν_Σ: $\Sigma \to \mathbf{M}_d(\Sigma)$ be a unital homomorphism. This map is completely determined by its values on the elements g_{ij} and it gives us the structure of a right Σ-module, in the free left Σ-module $\mathbb{V}_\Sigma \leftrightarrow \Sigma \otimes \mathbb{V}$, so that we have a bimodule structure. The right Σ-multiplication is given by

$$\theta_i q = \sum_j \nu_\Sigma(q)_{ij}\theta_j, \qquad \nu_\Sigma\colon \quad \mathbb{V} \otimes \Sigma \to \Sigma \otimes \mathbb{V},$$

$$\nu_\Sigma(\theta_i \otimes q) = \sum_j \nu_\Sigma(q)_{ij} \otimes \theta_j$$

This twisting preserves the product and the unit in Σ. The following condition completely fixes ν_Σ: $\nu_\Sigma(\mathrm{id} \otimes g) = (g \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma^{-1} \otimes \mathrm{id})$.

Lemma 2.1. The *-involutions on $\mathbb{V}$ and Σ naturally combine to a *-structure on the bimodule $\mathbb{V}_\Sigma$. The map ν_Σ: $\mathbb{V} \otimes \Sigma \to \Sigma \otimes \mathbb{V}$ is invertible and

$$*\nu_{\Sigma^*} = \nu_\Sigma^{-1} \tag{2}$$

It follows that $\mathbb{V}_\Sigma$ is free, as a right Σ-module.

Proof. The map $*$: $\mathbb{V}_\Sigma \to \mathbb{V}_\Sigma$ is introduced by $(q \otimes \theta_j)^* = \theta_j^* q^*$. It is sufficient to prove that such a map is involutive. This follows from (2) and the reality properties for σ and g. ∎

As a right/left Σ-module,

$$\mathbb{V}_\Sigma^{\otimes n} \leftrightarrow \underbrace{\mathbb{V} \otimes \cdots \otimes \mathbb{V}}_{n} \otimes \Sigma \leftrightarrow \Sigma \otimes \underbrace{\mathbb{V} \otimes \cdots \otimes \mathbb{V}}_{n}$$

Our definition of the Σ-bimodule structure implies that the braiding σ: $\mathbb{V} \otimes \mathbb{V} \to \mathbb{V} \otimes \mathbb{V}$ is (necessarily uniquely) extendible to a bimodule homomorphism σ: $\mathbb{V}_\Sigma^{\otimes 2} \to \mathbb{V}_\Sigma^{\otimes 2}$. Similarly, property (1) ensures that the metric tensor g: $\mathbb{V} \otimes \mathbb{V} \to \Sigma$ is uniquely extendible to a Σ-bilinear map g: $\mathbb{V}_\Sigma^{\otimes 2} \to \Sigma$.

Lemma 2.2. Let $\mathcal{M}$ and $\mathcal{N}$ be left Σ-modules and let Φ: $\mathcal{M} \to \mathcal{N}$ be a linear map satisfying $\Phi(g_{ij}\xi) = g_{ij}\Phi(\xi)$. Then Φ is left Σ-linear.

Proof. It is sufficient to check that Φ commutes with left multiplications by the matrix elements $[g^{-1}]_{ij}$. However this is equivalent to the above formula. ∎

Our extensions preserve all relevant algebraic relations between g, σ, and $*$. There is an interesting way to describe the relation between the left and the right Σ-module structures on $\mathbb{V}_\Sigma$,

$$(\mathrm{id} \otimes g)(\sigma \otimes \mathrm{id}) = (g \otimes \mathrm{id})(\mathrm{id} \otimes \sigma), \qquad \otimes \equiv \otimes_\Sigma \tag{3}$$

The space $\mathbb{V}_\Sigma$, together with the extended σ, generates a braided monoidal category $\mathscr{C}$. We shall use the same symbol σ to denote the generic braiding in this category and the name symbol g for extended contraction maps $g\colon \mathbb{V}_\Sigma^{\otimes n} \otimes_\Sigma \mathbb{V}_\Sigma^{\otimes n} \to \Sigma$, defined inductively by

$$x, y \in \mathbb{V}, \quad g\{(\psi \otimes x) \otimes (y \otimes \xi)\} = g(\psi, g(x, y)\xi)$$

where we shall also assume that tensors with different grades are mutually 'orthogonal'.

Consider a map $\langle\ \rangle\colon \mathbb{V}_\Sigma \times \mathbb{V}_\Sigma \to \Sigma$, $\langle\psi, \xi\rangle = g(\psi^*, \xi)$. Then

$$\langle\psi, \xi a\rangle = \langle\psi, \xi\rangle a, \qquad \langle\psi a, \xi\rangle = a^*\langle\psi, \xi\rangle, \qquad \langle\psi, a\xi\rangle = \langle a^*\psi, \xi\rangle$$

$$\langle\psi, \xi\rangle^* = \langle\xi, \psi\rangle, \qquad \langle\psi, \xi + \varphi\rangle = \langle\psi, \xi\rangle + \langle\psi, \varphi\rangle$$

This map plays the role of a Hermitian scalar product in $\mathbb{V}_\Sigma$.

(vii) *Strict positivity.* Assuming that Σ is realized in $H = l^2(\mathbb{Z})$, we have

$$\langle\xi, \xi\rangle = g(\xi^*, \xi) > 0, \qquad \forall \xi \in \mathbb{V}_\Sigma \backslash \{0\}$$

This condition is, in general, stronger than (iv). It is easy to construct examples where (iv) holds and (vii) fails. The above-introduced Σ-valued scalar product is naturally extendible to higher order tensor blocks $\mathbb{V}_\Sigma^{\otimes n}$. All algebraic properties are preserved.

In what follows, it will be assumed that conditions (i)–(iii) and (v)–(vii) are satisfied. The space $\mathbb{V}_\Sigma$ equipped with $\langle\ \rangle$ gives us a generally unbounded unitary bimodule over Σ. Furthermore, the extended scalar products on spaces $\mathbb{V}_\Sigma^{\otimes n}$ are understandable as n-fold tensor iterations of the initial bimodule $\mathbb{V}_\Sigma$. In particular, it follows that all extended $\langle\ \rangle$ are strictly positive, too (all $\mathbb{V}_\Sigma^{\otimes n}$ are unitary bimodules).

Lemma 2.3. The braid operator $\sigma\colon \mathbb{V}_\Sigma^{\otimes 2} \to \mathbb{V}_\Sigma^{\otimes 2}$ is Hermitian,

$$\langle\psi, \sigma(\xi)\rangle = \langle\sigma(\psi), \xi\rangle, \qquad \forall \psi, \xi \in \mathbb{V}_\Sigma^{\otimes 2}$$

The map $\sigma\colon \mathbb{V} \otimes \mathbb{V} \to \mathbb{V} \otimes \mathbb{V}$ is diagonalizable, and has real eigenvalues.

Proof: The Hermicity property follows from $*\sigma = \sigma*$ and

$$g(\psi, \sigma[\xi]) = g(\sigma[\psi], \xi), \qquad \forall \psi, \xi \in \mathbb{V} \otimes \mathbb{V}$$

which, in turn, follows from the definition of the bimodule structure on $\mathbb{V}_\Sigma$. Let $\omega: \Sigma \to \mathbb{C}$ be an arbitrary faithful state on Σ. Then $\omega \langle\,\rangle$ is a scalar product on $\mathbb{V} \otimes \mathbb{V}$, and σ is Hermitian with respect to this scalar product. ∎

3. BRAIDED EXTERIOR ALGEBRAS AND CLIFFORDIZATION

We follow Đurđevich and Oziewicz (1996), which means that braided Clifford algebras will be the Chevalley–Kähler deformations of braided exterior algebras. Let $\mathbb{V}^\wedge$ be the braided exterior algebra (Woronowicz, 1989) built over $(\sigma, \mathbb{V})$. This algebra is defined as $\mathbb{V}^\wedge = \mathbb{V}^\otimes/\ker(A_\sigma)$, where A_σ is the total braided antisymmetrizer map. The exterior algebra gets its *-structure from $\mathbb{V}^\otimes$. The algebra $\mathbb{V}^\wedge$ possesses a natural braided Hopf algebra structure, where the coproduct map is specified by

$$\phi(\alpha) = \sum_{i=0}^{n} B_{in-i}(\alpha), \qquad \alpha \in \mathbb{V}^{\wedge n}$$

and $B_{kl}: \mathbb{V}^{\wedge k+l} \to \mathbb{V}^{\wedge k} \otimes \mathbb{V}^{\wedge l}$ is the braided inverse-shuffle operator.

The coproduct map has a particularly elegant form if we make natural identifications induced by the antisymmetrizer map $\mathbb{V}^{\wedge n} \leftrightarrow \mathrm{im}(A_\sigma^n)$,

$$\alpha = \sum x_1 \otimes \cdots \otimes x_n, \qquad \phi(\alpha) = \sum_{i=0}^{n} \{x_1 \otimes \cdots \otimes x_i\} \otimes \{x_{i+1} \otimes \cdots \otimes x_n\}$$

The antipode map is braided-antimultiplicative (acting as a total σ-inverse permutation). The Hopf algebra structure is compatible with the *-involution, in the sense that

$$\phi(\alpha^*) = \phi(\alpha)^*, \qquad \kappa(\alpha^*) = \kappa(\alpha)^*$$

All considerations with the braided exterior algebra can be incorporated to the level of Σ-modules. Let $\mathbb{V}_\Sigma^\wedge$ be the braided exterior algebra constructed from $\mathbb{V}_\Sigma$ and the extended σ. Then we have the following natural identifications of left/right Σ-modules:

$$\mathbb{V}_\Sigma^\wedge \leftrightarrow \mathbb{V}^\wedge \otimes \Sigma \leftrightarrow \Sigma \otimes \mathbb{V}^\wedge$$

The coproduct map ϕ is naturally and uniquely extendible to a Σ-bilinear map $\phi: \mathbb{V}_\Sigma^\wedge \to \mathbb{V}_\Sigma^\wedge \otimes_\Sigma \mathbb{V}_\Sigma^\wedge$. In a similar way, it is possible to extend the coinverse and the counit.

The block antisymmetrizers $A_\sigma^n: \mathbb{V}^{\otimes n} \to \mathbb{V}^{\otimes n}$ are Hermitian maps and commute with the *-structure. Hence the pairing

$$g_\wedge: \quad \mathbb{V}_\Sigma^\otimes \times \mathbb{V}_\Sigma^\otimes \to \Sigma, \qquad g_\wedge(\psi, \xi) = g(\psi, A_\sigma \xi), \qquad g_\wedge(1, 1) = 1$$

is projectable down to a map $g_\wedge: \mathbb{V}_\Sigma^\wedge \times \mathbb{V}_\Sigma^\wedge \to \Sigma$.

Our braided Clifford algebra $\mathrm{cl}[\mathbb{V}, g, \sigma, \Sigma]$ is identified with $\mathbb{V}^{\wedge}_{\Sigma}$ at the level of Σ-bimodules. The *-structure will also be the same. However, $\mathrm{cl}[\mathbb{V}, g, \sigma, \Sigma]$ will be equipped with a new product defined by the Cliffordization (Rota and Stein, 1994),

$$\tilde{m} = m(\mathrm{id} \otimes g_{\wedge} \otimes \mathrm{id})(\phi \otimes \phi)$$

where $m: \mathbb{V}^{\wedge}_{\Sigma} \otimes \mathbb{V}^{\wedge}_{\Sigma} \to \mathbb{V}^{\wedge}_{\Sigma}$ is the original product in $\mathbb{V}^{\wedge}_{\Sigma}$. The first thing to examine is that we obtain a nice *-algebra this way:

Proposition 3.1. The product $\tilde{m}$ is associative and 1 is the unit element. The *-involution is $\tilde{m}$-antimultiplicative.

Proof. The associativity of the product follows from braided-multiplicativity of the coproduct, property (3), and the following interesting identities:

$$g_{\wedge}(m \otimes \mathrm{id}) = g_{\wedge}(\mathrm{id} \otimes g_{\wedge} \otimes \mathrm{id})(\mathrm{id} \otimes \phi)$$

$$g_{\wedge}(\mathrm{id} \otimes m) = g_{\wedge}(\mathrm{id} \otimes g_{\wedge} \otimes \mathrm{id})(\phi \otimes \mathrm{id})$$

The fact that 1 is the $\tilde{m}$-unit follows from $g_{\wedge}(1, \alpha) = g_{\wedge}(\alpha, 1) = \epsilon(\alpha)$. The $\tilde{m}$-antimultiplicativity of $*$ follows from standard commutation relations between $*$ and m, ϕ, $g_{\wedge}$. n

Definition 1. The constructed *-algebra $\mathrm{cl}[\mathbb{V}, g, \sigma, \Sigma]$ is called the braided Clifford algebra associated to $\mathbb{V}$, g, σ, and Σ.

4. EXTRA CONDITIONS

We are now going to discuss natural C*-type norms on $\mathrm{cl}[\mathbb{V}, g, \sigma, \Sigma]$. For this to work, it will be necessary to introduce a last set of our assumptions, regarding a more subtle behavior of σ.

A nice way to introduce such properties is to postulate the existence of an *auxiliary braid operator* $\tau: \mathbb{V}^{\otimes 2}_{\Sigma} \to \mathbb{V}^{\otimes 2}_{\Sigma}$, as will be discussed near the end of this paper. Playing with two braid operators will also enable us to prove interesting properties of σ and its braided exterior algebra $\mathbb{V}^{\wedge}_{\Sigma}$.

It is possible to proceed without making any extra assumptions on the existence and properties of τ; however, we have to postulate the positivity of braided antisymmetrizer maps.

(viii) *Positivity of braided antisymmetrizers*. All braided antisymmetrizer maps $A^n_{\alpha}: \mathbb{V}^{\otimes n}_{\Sigma} \to \mathbb{V}^{\otimes n}_{\Sigma}$ are positive operators. The positivity property is crucial to define a C*-algebraic structure on the Clifford algebra because only in this case will the scalar product $\langle\,\rangle_{\wedge}$ on $\mathbb{V}^{\wedge}_{\Sigma}$ given by the formula $\langle \alpha, \beta \rangle_{\wedge} = g_{\wedge}(\alpha^*, \beta)$ be strictly positive, giving us a structure of a generally unbounded unitary bimodule over Σ.

Proposition 4.1. The counit map ϵ: cl[$\mathbb{V}$, g, σ, Σ] $\to \Sigma$ is Hermitian, Σ-bilinear, and strictly positive.

Proof. Hermicity and Σ-linearity are obvious (the counit here is the projection on Σ). The strict positivity follows from the identity

$$\epsilon(\alpha^* \beta) = \langle \alpha, \beta \rangle_\wedge \quad \text{n} \tag{4}$$

When dealing with Hilbert space operators, there is an interesting assumption we can add to the list of properties of Σ—we can assume that the set of C*-algebraic norms on Σ distinguishes elements of Σ. Not every *-algebra possesses this property, and many *-algebras do not admit any representation by bounded operators. However, if Σ admits C*-algebraic norms, then they would be naturally extendible to cl[$\mathbb{V}$, g, σ, Σ].

To see this, we can consider the *left regular representation* of cl[$\mathbb{V}$, g, σ, Σ] in the Σ-bimodule $\mathbb{V}^{\wedge}_{\Sigma}$ According to (4), this representation is a *-representation,

$$\langle \alpha, T\beta \rangle_\wedge = \langle T^* \alpha, \beta \rangle_\wedge, \qquad \forall \alpha, \beta \in \mathbb{V}^{\wedge}_{\Sigma}, \quad \forall T \in \mathrm{cl}[\mathbb{V}, g, \sigma, \Sigma]$$

If in addition all the operators $T \in$ cl[$\mathbb{V}$, g, σ, Σ] are *continuous*, then there is a natural C*-norm on cl[$\mathbb{V}$, g, σ, Σ]. It is sufficient to check the continuity condition for generators from $\mathbb{V}$. If the braiding σ is such that there exists *a volume element* in $\mathbb{V}^\wedge$, then the continuity condition would hold automatically.

Let analyze a couple of cases when the positivity of antisymmetrizers would hold automatically. Assume that a self-adjoint braid operator τ: $\mathbb{V}^{\otimes 2} \to \mathbb{V}^{\otimes 2}$ is given, satisfying $*\tau* = \tau^{-1}$, extendible by Σ-linearity to $\mathbb{V}^{\otimes 2}_{\Sigma}$, and such that

$$\mathrm{im}(I - \sigma) = \ker(I + \tau), \qquad \mathrm{im}(I + \tau) = \ker(I - \sigma) \tag{5}$$

A consequence is that σ and τ commute and

$$\mathrm{im}(A^n_\Sigma) \subseteq \{\tau\text{-antisymmetric } n\text{-tensors}\} \tag{6}$$

This inclusion is a simple consequence of (5) and the fact that we can write

$$A^n_\sigma = [\mathrm{id}^k \otimes (I - \sigma) \otimes \mathrm{id}^{n-k-2}]T_k$$

where T_k: $\mathbb{V}^{\otimes n}_{\Sigma} \to \mathbb{V}^{\otimes n}_{\Sigma}$ is the part of the antisymmetrizer sum, containing permutations whose inverse does not reverse the order of $k + 1$ and $k + 2$.

Proposition 4.2. Assume that all σ-twists act in the same way on the vectors from the space of τ-antisymmetric n-tensors. Then, this space is σ-invariant.

If in addition $1 \in \mathbb{C}$ is the only positive eigeinvalue of σ, then all braided σ-antisymmetrizers will be positive.

Proof. The fact that σ-twists act in the same way on the τ-antisymmetric vectors means that we can always (trivial for $n > 4$; for $n = 2, 3, 4$, it is necessary to use the fact that σ and τ commute) exchange them with τ-twists (acting as -1). Hence, the space of τ-antisymmetric tensors is σ-invariant.

The second assumption means that $\sigma\colon \ker(I + \tau) \to \ker(I + \tau)$ is strictly negative, as the space $\ker(I + \tau)$ is spanned by all negative eigensubspaces of σ.

Therefore, all σ-antisymmetrizers are positive and their images coincide with τ-antisymmetric spaces. ∎

Proposition 4.3. Consider mutually equivalent properties

$$\begin{aligned}(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\sigma \otimes \mathrm{id}) &= (\mathrm{id} \otimes \sigma)(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \tau)\\ (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\tau \otimes \mathrm{id}) &= (\mathrm{id} \otimes \tau)(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)\end{aligned} \tag{7}$$

The above equations transform one to another by the *-conjugation. If they hold, and if τ-antisymmeric n-tensors are invariant under σ-twists, then all σ-twists act in the same way in this space.

Proof. Let us consider the case $n = 3$. If $\psi \in \mathbb{V}^{\otimes 3}$ is completely τ-antisymmetric and if the invariance property holds, then (7) gives $(\mathrm{id} \otimes \sigma)(\psi) = (\sigma \otimes \mathrm{id})(\psi)$. ∎

Furthermore, let us observe that the following pairs of strange equalities are equivalent

$$\begin{aligned}(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\tau \otimes \mathrm{id}) &= (\mathrm{id} \otimes \sigma)(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \tau)\\ (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\tau \otimes \mathrm{id}) &= (\mathrm{id} \otimes \tau)(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)\end{aligned} \tag{8}$$

$$\begin{aligned}(\mathrm{id} \otimes \tau)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \tau) &= (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\tau \otimes \mathrm{id})\\ (\mathrm{id} \otimes \tau)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \tau) &= (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\sigma \otimes \mathrm{id})\end{aligned} \tag{9}$$

The equivalent equalities are mutually adjoint. The equivalence also follows from the braid equation for τ and the fact that σ commutes with τ.

Lemma 4.4. If the above equalities hold, then the spaces of fully τ-antisymmetric tensors of order $n \geq 2$ are invariant under actions of all possible σ-twists.

Proof. The invariance under σ-twists easily follows from the commutation property, (8), and (9). ∎

Proposition 4.5. If the spaces of τ-antisymmetric operators are invariant under all possible σ-twists and if the restriction $\sigma\colon \ker(I + \tau) \to \ker(I + \tau)$ is negative, then all antisymmetrizer maps A_σ^n are positive. Furthermore, for each $n \geq 2$, we have

$$\mathrm{im}(A_\sigma^n) = \{\tau\text{-antisymmetric } n\text{-tensors}\} \tag{10}$$

Proof. Let $Y_n: \mathbb{V}_\Sigma^{\otimes n} \to \mathbb{V}_\Sigma^{\otimes n}$ be given by $Y_n = -\Sigma_{k=2}^n (-)^k \pi_{kn,\sigma}$, where the sum goes over permutations $\pi_{kn} \in S_n$ transposing 1 and the blocks $\{2, \ldots, k\}$ while acting trivially in $\{k+1, \ldots, n\}$. The braided antisymmetrizers satisfy the following recursive relations:

$$A_\sigma^{n+1} = \mathrm{id} \otimes A_\sigma^n - (\mathrm{id} \otimes Y_n)(\sigma \otimes A_\sigma^{n-1})(\mathrm{id} \otimes Y_n^\dagger) \tag{11}$$

Now, using induction on n, the negativity assumption for the restriction map $\sigma: \ker(I + \tau) \to \ker(I + \tau)$, and recursive formulas (11), it follows that restricted antisymmetrizers

$$A_\sigma^n: \quad \{\tau\text{-antisymm } n\text{-tensors}\} \to \{\tau\text{-antisymm } n\text{-tensors}\}$$

$$\{\tau\text{-antisymm } n\text{-tensors}\}^\perp = \sum_k \mathbb{V}_\Sigma^{\otimes k} \otimes \mathrm{im}(I + \tau) \otimes \mathbb{V}_\Sigma^{n-k-2} = \ker(A_\sigma^n)$$

are *strictly positive, in particular invertible* operators. In particular, (10) holds and obviously A_σ^n are positive everywhere. ∎

If (10) holds, it follows that the algebra $\mathbb{V}_\Sigma^\wedge$ is quadratic (generated by its quadratic relations). In this case, the Clifford algebra cl[$\mathbb{V}$, g, σ, Σ] can be viewed as the algebra built over $\mathbb{V}_\Sigma$ together with generating relations

$$\sum_\alpha x_\alpha y_\alpha = \sum_\alpha g(x_\alpha, y_\alpha), \qquad \textstyle\sum_\alpha \sigma(x_\alpha \otimes y_\alpha) = \sum_\alpha x_\alpha \otimes y_\alpha$$

We can *introduce spinors* as vectors of irreducible representations of the *-algebra cl[$\mathbb{V}$, g, σ, Σ] by bounded operators. Every such representation of cl[$\mathbb{V}$, g, σ, Σ] is (as generally for $\mathbb{C}$*-algebras) obtained from a pure state ω: cl[$\mathbb{V}$, g, σ, Σ] $\to \mathbb{C}$ via the GNS construction.

The algebra cl[$\mathbb{V}$, g, σ, Σ] may be infinite dimensional (the most interesting situations appear when Σ is infinite-dimensional) and possess nonequivalent irreducible representations.

In our context of framed quantum principal bundles (Đurđevich, 2000), the operator τ was coming from the bicovariant bimodule. In this context, it is natural to assume that Σ is of a 'bicovariant nature', too. Specifically, this requires the existence of a right $\mathcal{A}$-module structure $\circ: \Sigma \otimes \mathcal{A} \to \Sigma$ and a right $\mathcal{A}$-comodule structure $\varkappa_\Sigma: \Sigma \to \Sigma \otimes \mathcal{A}$ which is a (continuous) unital *-homomorphism and such that

$$(\alpha\beta) \circ a = (\alpha \circ a^{(1)})(\beta \circ a^{(2)}), \qquad 1 \circ a = \epsilon(a)1$$

$$\Sigma(q \circ a) = \sum_\alpha (q_\alpha \circ a^{(2)}) \otimes \kappa(a^{(1)}) c_\alpha a^{(3)}$$

Here $\mathcal{A}$ is the Hopf *-algebra corresponding to (polynomial functions over) a spinorial quantum structure group G. The maps $\circ$ and $\varkappa_\Sigma$ are completely fixed by postulating

$$g(x, y) \circ a = g(x \circ a^{(1)}, y \circ a^{(2)}), \qquad \varkappa_\Sigma g\,(x, y) = (g \otimes \mathrm{id})\varkappa(x \otimes y)$$

The above formulas extend to $\mathbb{V}_\Sigma^\wedge$.

ACKNOWLEDGMENT

I am very grateful to Prof. Zbigniew Oziewicz for kindly inviting me to take part in the Ixtapa Conference, and for numerous very interesting and fruitful discussions we have had on various occasions.

REFERENCES

Đurđevich, Micho (1996). Geometry of quantum principal bundles I, *Communications in Mathematical Physics*, **175**, 457–521.
Đurđevich, Micho (1997). Geometry of quantum principal bundles II, *Reviews in Mathemathical Physics*, **5**, 531–607.
Đurđevich, Micho (2000). Quantum spinor structures for quantum spaces, *International Journal of Theoretical Physics*, **40**, 115–138.
Đurđevich, Micho, and Zbigniew Oziewicz (1996). Clifford algebras and spinors for arbitrary braids, in: Julian Lawrynowicz, Editor, *Generalizations of Complex Analysis*, Banach Center Publications, Warsaw, pp. 315–325.
Lawrynowicz, Julian, L. C. Papaloucas, and Jakub Rembieliński (1994). Quantum braided Clifford algebras, in: Rafal Ablamowicz and Pertti Lounesto, Editors, *Clifford Algebras and Spinor Structures*, Kluwer, Dordrecht, pp. 387–395.
Oziewicz, Zbigniew (1997). Clifford algebra of multivectors, *Advances in Applied Clifford Algebras* **7**(S), 1–20.
Oziewicz, Zbigniew (1998). The Dirac operator as graph and the Clifford Hopf-gebra, in *Dirac Operators in Analysis*, Longman, Harlow, England, pp. 210–224.
Rota, Gian-Carlo, and J. A. Stein (1994). Plethystic Hopf algebras, *Proceedings of the National Academy of Sciences of the USA*, **91**, 13057–13061.
Woronowicz, Stanisław Lech (1987). Compact matrix pseudogroups, *Communications in Mathematical Physics*, **111**, 613–665.
Woronowicz, Stanisław Lech (1989). Differential calculus on compact matrix pseudogroups, *Communications in Mathematical Physics*, **122**, 125–170.